

Coordinate representation of particle dynamics in AdS and in generic static spacetimes

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Abstract

We discuss the quantum dynamics of a particle in static curved spacetimes in a coordinate representation. The scheme is based on the analysis of the squared energy operator E^2 , which is quadratic in momenta and contains a scalar curvature term. Our main emphasis is on AdS spaces, where this term is fixed by the isometry group. As a byproduct the isometry generators are constructed and the energy spectrum is reproduced. In the massless case the conformal symmetry is realized as well. We show the equivalence between this quantization and the covariant quantization, based on the Klein-Gordon type equation in AdS. We further demonstrate that the two quantization methods in an arbitrary $(N + 1)$ -dimensional static spacetime are equivalent to each other if the scalar curvature terms both in the operator E^2 and in the Klein-Gordon type equation have the same coefficient equal to $\frac{N-1}{4N}$.

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1 Introduction

In generic spacetimes the notion of a quantized particle becomes dependent on the local coordinate system [1]. Only in the presence of a high degree of symmetry one is able to define particles in a global covariant manner. In Anti-de Sitter space AdS_{N+1} , like in the most prominent case of Minkowski space $\mathbb{R}^{1,N}$, particles are identified with unitary irreducible representations of the corresponding isometry groups [2].

The aim of this paper is twofold. On one side we want to make a contribution to the study of the quantization problem for a relativistic particle in a generic static spacetime. There, at least, the spectral problem for the energy in certain suitably chosen coordinate systems is a reasonable one. Its study includes ordering problems. We will show that even in the absence of further symmetries, one nevertheless can fix the ordering ambiguity. For this purpose we insist on general covariance with respect to space and require equivalence of the quantization based on the Klein-Gordon equation to quantization after Hamiltonian reduction realized in a coordinate representation.

The second aim is the application of the coordinate representation, to be developed in the body of the paper, to the quantized particle in AdS_{N+1} . In this case the ordering ambiguity can be fixed exclusively within the coordinate representation by demanding that the algebra of the isometry generators $\mathfrak{so}(2, N)$ can be realized. In this way we will find a coordinate version of the unitary irreducible representations of $\text{SO}(2, N)$. Of course we will reproduce all the well-known results concerning the spectrum of the relevant quantum numbers, see e.g. [3]. With the AdS/CFT correspondence in mind we present this technique also because of its potential use for string quantization in AdS_{N+1} .

The paper is organized as follows. We first start with some introductory aspects of the coordinate representation in generic static spacetimes and then specialize to AdS_{N+1} with its $\text{SO}(2, N)$ representation. After this, still for AdS_{N+1} , a comparison with the spacetime covariant treatment based on the Klein-Gordon equation is made. Finally we come back to generic static spacetimes.

Let us consider a $(N + 1)$ -dimensional spacetime with coordinates x^μ , $\mu = (0, 1, \dots, N)$ and a static metric tensor

$$g_{\mu\nu} = \begin{pmatrix} g_{00}(x) & 0 \\ 0 & g_{mn}(x) \end{pmatrix}, \quad (1.1)$$

where $g_{00} = -e^f$ and g_{mn} are functions only of the spatial coordinates x^n ($n = 1, \dots, N$).

The dynamics of a particle in this background is described by the action

$$S = \int d\tau \left(p_\mu \dot{x}^\mu + \lambda (g^{\mu\nu} p_\mu p_\nu + M^2) \right), \quad (1.2)$$

and in the gauge $x^0 = \tau$ it reduces to an ordinary Hamiltonian system

$$S = \int d\tau \left(p_n \dot{x}^n - E(p, x) \right), \quad (1.3)$$

where $E(p, x) = -p_0 > 0$ is the particle energy.

From the mass shell condition $g^{\mu\nu} p_\mu p_\nu + M^2 = 0$ one obtains the squared energy

$$E^2 = e^{f(x)} g^{mn}(x) p_m p_n + M^2 e^{f(x)}. \quad (1.4)$$

This function can be associated with the Hamiltonian of a non-relativistic particle moving in the potential $M^2 e^{f(x)}$ in a curved background with the metric tensor

$$h_{mn}(x) = e^{-f(x)} g_{mn}(x). \quad (1.5)$$

Quantizing this system in the coordinate representation, one gets the Hilbert space with wave functions $\psi(x)$ and the scalar product

$$\langle \psi_2 | \psi_1 \rangle = \int d^N x \sqrt{h(x)} \psi_2^*(x) \psi_1(x), \quad (1.6)$$

where $h(x) = \det h_{mn}(x)$. The Hermitian momentum operators are $p_n = -i\partial_n - \frac{i}{4}\partial_n \log h$.

Since (1.4) is quadratic in momenta, the ordering ambiguities contain at most second derivatives of $h_{mn}(x)$. Requiring general covariance in the reduced manifold, the freedom due to ordering ambiguities is parameterized by a constant a in

$$E^2 = -\Delta_h + a\mathcal{R}_h(x) + M^2 e^{f(x)}, \quad (1.7)$$

where Δ_h is the covariant Laplace operator for the metric tensor h_{mn} , and \mathcal{R}_h denotes the corresponding scalar curvature [4] (for further references see [5]). One of our tasks is to fix the value of a .

2 Classical description of AdS particle

AdS_{N+1} is realized as a hyperboloid $X^A X_A = -R^2$, where X^A , $A = (0', 0, 1, \dots, N)$ are coordinates of the embedding space $\mathbb{R}^{2,N}$, which we parameterize by¹

$$X^{0'} = \frac{R \sin \theta}{\sqrt{1 - x^2}}, \quad X^0 = \frac{R \cos \theta}{\sqrt{1 - x^2}}, \quad X^n = \frac{R x_n}{\sqrt{1 - x^2}}, \quad (2.1)$$

¹For notational convenience we write the spatial coordinates with down indices.

with $x^2 := x_n x_n < 1$. The polar angle θ is interpreted as a dimensionless time coordinate $\theta = x^0$, since the induced metric tensor on the hyperboloid has the structure (1.1) with

$$g_{00} = -\frac{R^2}{1-x^2}, \quad g_{mn} = \frac{R^2}{1-x^2} \left(\delta_{mn} + \frac{x_m x_n}{1-x^2} \right). \quad (2.2)$$

If a vector field $\mathcal{V} = \mathcal{V}^\mu \partial_\mu$ generates spacetime isometry transformations, i.e. $\mathcal{L}_{\mathcal{V}} g_{\mu\nu} = 0$, then $J = \mathcal{V}^\mu p_\mu$ is a Noether integral of the system (1.2), and in the gauge $x^0 = \tau$ it becomes $J = \mathcal{V}^n(\tau, x) p_n - \mathcal{V}^0(\tau, x) E(p, x)$, where E is the particle energy obtained from (1.4).

The isometries of AdS_{N+1} are the $\text{SO}(2, N)$ transformations generated by the vector fields \mathcal{V}_{AB} , whose action on the embedding space coordinates is $\mathcal{V}_{AB}(X^C) = \delta_A^C X_B - \delta_B^C X_A$. By (2.1) we then find

$$\begin{aligned} \mathcal{V}_{0'0} &= -\partial_\theta, & \mathcal{V}_{mn} &= x_n \partial_m - x_m \partial_n, \\ \mathcal{V}_{n0'} &= -x_n \cos \theta \partial_\theta - \sin \theta V_n, & \mathcal{V}_{n0} &= x_n \sin \theta \partial_\theta - \cos \theta V_n, \end{aligned} \quad (2.3)$$

where $V_n = \partial_n - x_n D$ and D is the dilatation operator $D = x_m \partial_m$. The corresponding Noether integrals are given by (again in the gauge $\tau = x^0 = \theta$)

$$J_{0'0} = E = \sqrt{p^2 - (p \cdot x)^2 + \frac{M^2 R^2}{1-x^2}}, \quad J_{mn} = p_m x_n - p_n x_m, \quad (2.4)$$

$$J_{n0'} = E x_n \cos \tau + ((p \cdot x)x_n - p_n) \sin \tau, \quad J_{n0} = ((p \cdot x)x_n - p_n) \cos \tau - E x_n \sin \tau. \quad (2.5)$$

Note that the Casimir invariant of these integrals is constant $\frac{1}{2} J_{AB} J^{AB} = M^2 R^2$ and the minimal value of energy E_0 is equal to² $E_0 = MR$.

At $\tau = 0$, the boost generators reduce to

$$J_{n0'} = E x_n, \quad J_{n0} = (p \cdot x)x_n - p_n. \quad (2.6)$$

We also use their complex combinations

$$Z_n = J_{n0'} - i J_{n0}, \quad Z_n^* = J_{n0'} + i J_{n0}, \quad (2.7)$$

which in quantum theory become lowering and raising operators for the energy spectrum.

The Poisson brackets of the functions (2.4), (2.6) form the $\mathfrak{so}(2, N)$ algebra. Its compact part $\mathfrak{so}(2) \oplus \mathfrak{so}(N)$ is trivially realized by E and the rotation generators J_{mn} , and the rest of the algebra can be written as

$$\{E, Z_n\} = -i Z_n, \quad \{Z_m, Z_n\} = 0, \quad \{Z_m, Z_n^*\} = 2i \delta_{mn} E - 2J_{mn}, \quad (2.8)$$

$$\{J_{lm}, Z_n\} = \delta_{ln} Z_m - \delta_{mn} Z_l. \quad (2.9)$$

The squared energy in AdS is given by

$$E^2 = h^{mn}(x) p_m p_n + \frac{M^2 R^2}{1-x^2}, \quad (2.10)$$

²More precisely, E is the particle energy measured in units of $1/R$.

with $h^{mn}(x) = \delta_{mn} - x_m x_n$. Its inverse is the matrix

$$h_{mn}(x) = \delta_{mn} + \frac{x_m x_n}{1 - x^2}, \quad \text{and} \quad h = \frac{1}{1 - x^2}. \quad (2.11)$$

It is just the metric tensor of the N -dimensional unit semi-sphere.

Concluding this section we consider the massless particle in AdS. For $M = 0$ the action (1.2) is invariant under conformal transformations. In AdS_{N+1} those, which are not isometries, are generated by the conformal Killing vectors \mathcal{V}_A with the property (see for example [6])³

$$\mathcal{V}_A(X^B) = R \delta_A^B + \frac{X_A X^B}{R}, \quad \mathcal{L}_{\mathcal{V}_A} g_{\mu\nu} = \frac{2X_A}{R} g_{\mu\nu}. \quad (2.12)$$

Then, from (2.1) one gets

$$\begin{aligned} \mathcal{V}_{0'} &= \sqrt{1 - x^2} (\cos \theta \partial_\theta - \sin \theta D), \quad \mathcal{V}_0 = -\sqrt{1 - x^2} (\sin \theta \partial_\theta + \cos \theta D), \\ \mathcal{V}_n &= \sqrt{1 - x^2} \partial_n. \end{aligned} \quad (2.13)$$

The corresponding dynamical integrals at $\tau = 0$ become

$$C_{0'} = -\sqrt{1 - x^2} E, \quad C_0 = -\sqrt{1 - x^2} (p \cdot x), \quad C_n = \sqrt{1 - x^2} p_n, \quad (2.14)$$

where E is the energy of the massless particle, with

$$E^2 = p^2 - (p \cdot x)^2. \quad (2.15)$$

This E^2 coincides with the Hamiltonian of a free particle on the unit semi-sphere. That is why the massless particle reaches the boundary in a finite time.

3 AdS particle in coordinate representation

A consistent quantization of the particle dynamics in AdS_{N+1} should provide a unitary irreducible representation of the $\text{SO}(2, N)$ group. We use the classical expressions of dynamical integrals (2.4), (2.6) and apply the coordinate representation. To specify the ordering prescription for the operators we first consider E^2 . Since the scalar curvature of the unit sphere is equal to $N(N - 1)$, the prescription (1.7) provides the operator

$$E^2 = -\sqrt{1 - x^2} \partial_m \frac{\delta_{mn} - x_m x_n}{\sqrt{1 - x^2}} \partial_n + a N(N - 1) + \frac{M^2 R^2}{1 - x^2}. \quad (3.1)$$

We will fix the parameter a below. Since E^2 will turn out to be positive, its positive square root defines the energy operator E .

The rotation generators in (2.4) have no ordering ambiguity and they take the standard form

$$J_{mn} = i(x_m \partial_n - x_n \partial_m). \quad (3.2)$$

³As usual, the case AdS_2 is special and has to be treated separately.

For the ordering in the boost generators $J_{n0'}$ (2.6) we guess

$$J_{n0'} = \sqrt{E} x_n \sqrt{E} , \quad (3.3)$$

and get the commutator

$$[E^2, J_{n0'}] = \sqrt{E} \left(N x_n - 2V_n \right) \sqrt{E} . \quad (3.4)$$

On the other hand the $\mathfrak{so}(2, N)$ algebra requires

$$[E^2, J_{n0'}] = 2i J_{n0} E + J_{n0'} . \quad (3.5)$$

Comparing these two commutators we obtain

$$J_{n0} = i\sqrt{E} \left(V_n - \frac{N-1}{2} x_n \right) \frac{1}{\sqrt{E}} . \quad (3.6)$$

By (3.1), (3.3) and (3.6) we can verify the commutator similar to (3.5)

$$[E^2, J_{n0}] = -2i J_{n0'} E + J_{n0} . \quad (3.7)$$

This condition is fulfilled if the constant term in E^2 is equal to $\frac{(N-1)^2}{4}$. That fixes the coefficient of the scalar curvature

$$a = \frac{N-1}{4N} . \quad (3.8)$$

Having fixed the ordering ambiguities by requiring the preservation of part of the consequences of the $\mathfrak{so}(2, N)$ algebra, we now discuss the issue of energy eigenfunctions and spectrum. Our Hilbert space is given by the wave functions $\Psi(x)$ with the scalar product

$$\langle \Psi_2 | \Psi_1 \rangle = \int_{x^2 < 1} \frac{d^N x}{\sqrt{1-x^2}} \Psi_2^*(x) \Psi_1(x) . \quad (3.9)$$

The ground state has to be a $\text{SO}(N)$ scalar function, which is annihilated by the lowering operators $Z_n = J_{n0'} - iJ_{n0}$. These conditions are equivalent to the equations

$$\left(x_n \left(E_0 - \frac{N-1}{2} \right) + V_n \right) \Psi_{E_0}(x^2) = 0 , \quad (3.10)$$

where E_0 denotes the energy of the ground state. This yields the wave function

$$\Psi_{E_0} \sim (1-x^2)^{\frac{E_0}{2} - \frac{N-1}{4}} . \quad (3.11)$$

Since (3.11) should be an eigenfunction of the operator (3.1) with the eigenvalue E_0^2 , we can relate the mass to the lowest energy value by

$$M^2 R^2 = \left(E_0 - \frac{N}{2} \right)^2 - \frac{1}{4} . \quad (3.12)$$

The finiteness of the norm of the vacuum wave function (3.11) reproduces the well known unitarity bound $E_0 > \frac{N}{2} - 1$ [7]. From (3.12) then follows that the two values of E_0

$$E_0^\pm = \frac{N}{2} \pm \sqrt{M^2 R^2 + \frac{1}{4}}, \quad (3.13)$$

correspond to the same M^2 , if $-\frac{1}{4} \leq M^2 R^2 < \frac{3}{4}$, and for $M^2 R^2 \geq \frac{3}{4}$ only E_0^+ is admissible.

The action of the raising operators Z_n^* on the vacuum state creates higher level eigenfunctions. However, already on the second level the states $Z_m^* Z_n^* \Psi_{E_0}$ with $m \neq n$ are not orthogonal to each other. Lower dimensional cases are exceptional and they are presented in the Appendix. To find orthogonal eigenfunctions in higher dimensional cases we use the Casimir operator of the rotation generators $L^2 = \frac{1}{2} J_{mn} J_{mn}$ and rewrite the operator (3.1) with (3.8) and (3.12) as

$$E^2 = -4z(1-z)\partial_z^2 + 2((N+1)z - N)\partial_z + \frac{1}{z}L^2 + \frac{(N-1)^2}{4} + \frac{(E_0 - \frac{N}{2})^2 - \frac{1}{4}}{1-z}, \quad (3.14)$$

with $z = x^2$. We look for the eigenfunctions of these operator in the form $\Psi = F(z) Y_L(\Omega)$, where $Y_L(\Omega)$ is a spherical harmonic, which is an eigenfunction of the operator L^2 with the eigenvalue $L(L+N-2)$. Then, the eigenvalue problem

$$E^2 \Psi_\omega = \omega^2 \Psi_\omega \quad (3.15)$$

is solved by

$$F(z) = z^{\frac{L}{2}} (1-z)^{\frac{E_0 - \frac{N-1}{2}}{2}} {}_2F_1(a, b, c; z), \quad (3.16)$$

where ${}_2F_1(a, b, c; z)$ is the hypergeometric function with the parameters

$$a = \frac{1}{2}(E_0 + L - \omega), \quad b = \frac{1}{2}(E_0 + L + \omega), \quad c = \frac{N}{2} + L. \quad (3.17)$$

The regularity condition at $z = 1$ requires $a = -n$ and one gets the energy spectrum (in agreement with [2, 7, 3])

$$\omega_{n,L} = E_0 + L + 2n. \quad (3.18)$$

It coincides with the spectrum of the N -dimensional normal ordered and shifted (by E_0) harmonic oscillator Hamiltonian $\hat{H} = a_n^* a_n + E_0$ [8].

It remains to check the full set of commutation relations between the symmetry generators. Nontrivial are only the commutators corresponding to (2.8). First note that from (3.5) and (3.7) follows the commutator $[E^2, Z_n^*] = Z_n^* (1 + 2E)$, which is equivalent to the statement: if Ψ_ω is an eigenstate of E^2 with the eigenvalue ω^2 , then $Z_n^* \Psi_\omega$ is also an eigenstate of E^2 with the eigenvalue $(\omega + 1)^2$. This implies $[E, Z_n^*] = Z_n^*$. Now, only the commutation relations between the boost operators need discussion.

Calculating the commutator $[J_{m0'}, J_{n0'}]$, we rewrite it in the form

$$[J_{m0'}, J_{n0'}] = \sqrt{E} (x_m E x_n E - x_n E x_m E) \frac{1}{\sqrt{E}}, \quad (3.19)$$

and use the relation

$$E x_n E - x_n E^2 = \frac{N-1}{2} x_n - V_n , \quad (3.20)$$

which easily follows from the commutator $[E, J_{n0}] = iJ_{n0}$. With the help of (3.20), we find that the operator in the parentheses in (3.19) is J_{mn} . Since $[E, J_{mn}] = 0$, we can neglect the operators \sqrt{E} in (3.19) and obtain $[J_{m0'}, J_{n0'}] = iJ_{mn}$. The other commutation relations of the boost generators are derived in a similar way. In particular, from $[E, J_{n0}] = -iJ_{n0'}$ follows

$$E V_n E - V_n E^2 = \left(\frac{(N-1)^2}{4} - E^2 \right) x_n - \frac{N-1}{2} V_n , \quad (3.21)$$

which together with (3.20) is helpful to check the commutator $[J_{m0}, J_{n0'}] = i\delta_{mn}E$.

The calculation of the Casimir operator can be done also with the help of (3.20) and it yields $E_0(E_0 - N)$. Note that the relation of the Casimir to the lowest energy value has been renormalized relative to the classical expression E_0^2 .

Finally, we discuss the massless case. According to (3.12) it corresponds to

$$E_0^\pm = \frac{N \pm 1}{2} , \quad (3.22)$$

with the ground state wave functions $\Psi_{E_0^-} \sim 1$, $\Psi_{E_0^+} \sim \sqrt{1-x^2}$.

To construct the generators of conformal transformations we use the functions (2.14) and first introduce the operator $C_{0'}$ similarly to the boost generators (3.3). Then, C_0 and C_n can be defined by the commutators of $C_{0'}$ with E^2 and $J_{n0'}$, respectively. As a result we find

$$\begin{aligned} C_{0'} &= -\sqrt{E} \sqrt{1-x^2} \sqrt{E} , & C_0 &= i\sqrt{E} \sqrt{1-x^2} \left(D + \frac{N-1}{2} \right) \frac{1}{\sqrt{E}} , \\ C_n &= -i\sqrt{E} \sqrt{1-x^2} \partial_n \frac{1}{\sqrt{E}} . \end{aligned} \quad (3.23)$$

These operators together with the isometry generators realize the commutation relations of the conformal group $SO(2, N+1)$. Like for the isometry generators, the check goes as follows: one first realizes the $\mathfrak{sl}(2)$ algebra with E , $C_{0'}$, C_0 and then derives operator identities similar to (3.20), (3.21), which become helpful to verify the other commutation relations. Introducing the lowering and raising operators $Z = C_{0'} - iC_0$ and $Z^* = C_{0'} + iC_0$, one finds

$$Z\Psi_{E_0^+} \sim \Psi_{E_0^-} , \quad Z^*\Psi_{E_0^-} \sim \Psi_{E_0^+} . \quad (3.24)$$

Therefore, the conformal symmetry is realized on the states constructed by the action of the creation operators (Z_n^*, Z^*) on both ground states $\Psi_{E_0^-}$ and $\Psi_{E_0^+}$ [6].

For AdS_2 note that $E_0^- = 0$ and the operator $1/\sqrt{E}$ becomes singular. That also makes the discussion of the conformal symmetry in AdS_2 special.

4 Covariant quantization in AdS

Covariant quantization of the particle dynamics in AdS_{N+1} is based on the analysis of the Klein-Gordon type equation

$$(\square - \mathcal{M}^2)\Phi = 0 , \quad (4.1)$$

where \square is the covariant d'Alembertian in AdS_{N+1} . Eq. (4.1) is a quantum analog of the mass shell condition $g^{\mu\nu} p_\mu p_\nu + M^2 = 0$ and, in general, one has to make the replacement

$$g^{\mu\nu} p_\mu p_\nu \mapsto -\square - \tilde{a} \mathcal{R}_g , \quad (4.2)$$

where \mathcal{R}_g is the spacetime scalar curvature and \tilde{a} is a constant. Since \mathcal{R}_g is constant for AdS_{N+1} and equal to $-\frac{N(N+1)}{R^2}$, this ambiguity has been included in \mathcal{M}^2

$$\mathcal{M}^2 = M^2 - \tilde{a} \frac{N(N+1)}{R^2} . \quad (4.3)$$

Our task is now to fix the ambiguity of the value of \mathcal{M}^2 by requiring exact correspondence to the previous treatment in coordinate representation.

The Hilbert space of the covariant quantization is formed by the positive frequency solutions of the equation (4.1)

$$\Phi_\omega(\theta, x) = e^{-i\omega\theta} \phi_\omega(x) , \quad \omega > 0 , \quad (4.4)$$

with the standard scalar product

$$\langle \Phi_2 | \Phi_1 \rangle = \frac{i}{2} \int_{\theta=\tau} d^N x \sqrt{-g} g^{00} (\partial_\theta \Phi_2^* \Phi_1 - \Phi_2^* \partial_\theta \Phi_1) . \quad (4.5)$$

Using again the coordinates $x^\mu = (\theta, x_n)$, by (2.2) we find

$$g = -\frac{R^{2N+2}}{(1-x^2)^{N+2}} , \quad g^{mn} = \frac{1-x^2}{R^2} (\delta_{mn} - x_m x_n) , \quad (4.6)$$

and the equation for $\phi_\omega(x)$ obtained from (4.1) and (4.4) takes the form of an eigenvalue problem

$$-(1-x^2)^{N/2} \partial_m \frac{\delta_{mn} - x_m x_n}{(1-x^2)^{N/2}} \partial_n \phi_\omega(x) + \frac{\mathcal{M}^2 R^2}{1-x^2} \phi_\omega(x) = \omega^2 \phi_\omega(x) . \quad (4.7)$$

If we multiply this equation by $(1-x^2)^{\frac{1-N}{4}}$ and make the replacements

$$\mathcal{M}^2 R^2 \mapsto E_0(E_0 - N) , \quad \phi_\omega(x) \mapsto (1-x^2)^{\frac{N-1}{4}} \Psi_\omega(x) , \quad (4.8)$$

we get just the eigenvalue problem (3.15) for the operator (3.14).

To justify these replacements and find the exact correspondence between the two quantization methods we compare the corresponding scalar products and the symmetry generators.

When two wave functions (4.4) have equal ω , the scalar product (4.5) reduces to

$$\langle \Phi_2 | \Phi_1 \rangle = R^{N-1} \int \frac{d^N x}{(1-x^2)^{N/2}} \phi_{\omega,2}^*(x) \omega \phi_{\omega,1}(x) . \quad (4.9)$$

The local integration measure here differs from $\sqrt{h(x)}$ just by the factor compensated in the rescaling (4.8). The exact correspondence between the wave functions then takes the form

$$\Phi_\omega(\theta, x) = \frac{e^{-i\omega\theta}}{\sqrt{R^{N-1} \omega}} (1-x^2)^{\frac{N-1}{4}} \Psi_\omega(x) , \quad (4.10)$$

where ω is an eigenvalue of the operator E . For generic superpositions of energy eigenfunctions the correspondence (4.10) implies

$$\begin{aligned}\phi(x) &= \int_{y^2 < 1} \frac{d^N y}{\sqrt{1-y^2}} K(x, y) \Psi(y) , \\ K(x, y) &= \sum_{L, l, n} \frac{(1-x^2)^{\frac{N-1}{4}}}{\sqrt{R^{N-1} \omega_{n, L}}} \Psi_{n, L, l}(x) \Psi_{n, L, l}^*(y) ,\end{aligned}\tag{4.11}$$

with l a collective (angular momentum) index counting the degeneracy of the energy levels (3.18). The function $K(x, y)$ is the AdS analog of the Newton-Wigner function [10] in Minkowski space and $\phi(x) = \Phi(0, x)$.

The wave functions $\Phi(\theta, x)$ are scalar fields under the isometry transformations. This implies that the symmetry generators of the covariant quantization are given by

$$J_{AB} = -i \mathcal{V}_{AB} ,\tag{4.12}$$

where \mathcal{V}_{AB} are the vector fields of the isometry transformations (2.3). The relation between the Casimir operator of the generators (4.12) and the covariant d'Alembertian

$$\frac{1}{2} J_{AB} J^{AB} = R^2 \square ,\tag{4.13}$$

justifies the replacement $\mathcal{M}^2 R^2 \rightarrow E_0(E_0 - N)$ in (4.7).

Note that this expression for \mathcal{M}^2 together with (3.12) and (4.3) fixes also $\tilde{a} = \frac{N-1}{4N}$. The bound $M^2 R^2 \geq -\frac{1}{4}$ stated after eq. (3.13) then corresponds to the Breitenlohner-Freedman bound $\mathcal{M}^2 R^2 \geq -\frac{N^2}{4}$ [7].

Let us now consider the relation between the symmetry generators (4.12) and their counterparts in the coordinate representation. From the correspondence (4.10) we find that the operators (4.12) are mapped just to the operators constructed in the previous section. The rule for this map is the following: one has to replace the operator $i\partial_\theta$ by E , then set $\theta = 0$ in (4.12), (2.3) and then multiply the obtained operators by $\sqrt{E} (1-x^2)^{\frac{1-N}{4}}$ from left and by $(1-x^2)^{\frac{N-1}{4}}/\sqrt{E}$ from right.

A last comment concerns the massless case. In the covariant approach it is described by the equation

$$\left(\square + \frac{N^2 - 1}{4R^2} \right) \Phi = 0 ,\tag{4.14}$$

corresponding to $E_0^\pm = \frac{N \pm 1}{2}$. This equation is invariant under conformal transformations of the wave function Φ that infinitesimally can be written as [11] (see (2.12))

$$\Phi \mapsto \Phi + \epsilon^A \left(\mathcal{V}_A + \frac{N-1}{2R} X_A \right) \Phi ,\tag{4.15}$$

where X_A are the embedding coordinates and \mathcal{V}_A are the vector fields of the conformal transformations (2.13). One can check that the above described correspondence between the operators of the covariant quantization and the coordinate representation matches the generators of the conformal transformations as well.

5 Particle in a static spacetime

At the end we return to the general case started with the operator (1.7). Its eigenvalue problem is equivalent to the equation

$$\Delta_h \psi_\omega(x) + \left(\omega^2 - a \mathcal{R}_h(x) - M^2 e^{f(x)} \right) \psi_\omega(x) = 0 . \quad (5.1)$$

According to (1.5) h_{mn} is a rescaling of \hat{g}_{mn} , the spatial part of the metric tensor. Therefore, the scalar curvature here can be written as

$$\mathcal{R}_h = e^f \left(\mathcal{R}_{\hat{g}} + (N-1) \Delta_{\hat{g}} f - \frac{(N-1)(N-2)}{4} g^{mn} \partial_m f \partial_n f \right) . \quad (5.2)$$

The Laplace operators of the rescaled metrics are related by

$$\Delta_h = e^f \left(\Delta_{\hat{g}} - \left(\frac{N}{2} - 1 \right) g^{mn} \partial_m f \partial_n f \right) . \quad (5.3)$$

Now we consider the covariant quantization of the same system, based on the Klein-Gordon type equation with the replacement rule (4.2)

$$(\square - \tilde{a} \mathcal{R}_g - \tilde{M}^2) \Phi = 0 . \quad (5.4)$$

The positive frequency solutions of these equation $\Phi = e^{-i\omega x^0} \phi_\omega(x)$ form a Hilbert space with the scalar product (4.5) and the wave function $\phi_\omega(x)$ satisfies the equation

$$\sqrt{\frac{e^f}{\hat{g}}} \partial_m \sqrt{e^f \hat{g}} g^{mn} \partial_n \phi_\omega(x) + \left(\omega^2 - \tilde{a} e^f \mathcal{R}_g - \tilde{M}^2 e^f \right) \phi_\omega(x) = 0 , \quad (5.5)$$

that can also be treated as an eigenvalue problem. To relate it with (5.1), we introduce the map between the eigenfunctions

$$\phi_\omega(x) = \frac{e^{\frac{1-N}{4} f(x)}}{\sqrt{\omega}} \psi_\omega(x) , \quad (5.6)$$

compatible with the scalar products (1.6), (4.5) and (4.9).

Using the relation between the scalar curvatures \mathcal{R}_g and $\mathcal{R}_{\hat{g}}$

$$\mathcal{R}_g = \mathcal{R}_{\hat{g}} - \Delta_{\hat{g}} f - \frac{1}{2} g^{mn} \partial_m f \partial_n f , \quad (5.7)$$

and eqs. (5.2)-(5.3), we find that (5.1) and (5.5) are equivalent to each other for

$$\tilde{M} = M , \quad \tilde{a} = a = \frac{N-1}{4N} . \quad (5.8)$$

This value of the coefficient \tilde{a} corresponds to the Weyl invariance for $M = 0$ (see e.g. [11]).

6 Conclusions

We discussed the quantization of a particle in $(N + 1)$ -dimensional static spacetimes after Hamiltonian reduction within a coordinate representation. The key point of this analysis is the observation, that the expression for the squared energy looks like the Hamiltonian for a non-relativistic particle in a curved N -dimensional space. This fact has been used in the literature at several places, see e.g. [12, 13, 14]. However, the use of this observation for fixing some ordering ambiguities and for a complete realization of symmetry algebras in terms of operators acting on position space wave functions is a new facet of this issue. The description of a quantum particle in terms of position dependent wave functions seems to be most naturally. But in order to respect the relativistic principles, the generators for transformations involving time (energy, boosts) become non-local.

The ordering ambiguities concern the value of a factor in front of the scalar curvature which has been fixed to $a = \frac{N-1}{4N}$. Applying in different situations other principles or normalization conditions, one generates other values for a , compare e.g. the value appearing in the setting of [15, 16, 5]. For the isometry algebra of AdS_{N+1} we found a new representation in terms of operators acting on functions depending on N -dimensional space coordinates. To get this representation, we had to solve further ordering problems for all the other generators besides the squared energy.

Usually one notes a posteriori that the Klein-Gordon equation for $\mathcal{M}^2 = M_w^2 = -\frac{N^2-1}{4R^2}$ exhibits conformal invariance [17, 9] and therefore $M^2 = \mathcal{M}^2 - M_w^2$ should be called *the* squared mass. Our analysis sheds some different light on the interpretation of the mass parameter \mathcal{M}^2 in the AdS_{N+1} Klein-Gordon equation. We use no field theoretical arguments. On the basis of pure particle quantum mechanics our M^2 is a priori the squared mass, and the shift to \mathcal{M}^2 is generated by the quantum mechanical ordering effect.

As a byproduct we constructed the AdS analog of the Minkowski space Newton-Wigner wave functions [10]. It would be interesting to perform the sum explicitly and to compare its analytic properties to the Minkowski case and recent discussion in de Sitter space [18]. Note that a generalization of our quantization scheme to non-static spacetimes with a global time coordinate is straightforward.

Furthermore, one could try to apply the techniques of our paper to the quantization of strings in AdS using a gauge in which the energy density along the string is constant.

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A Low dimensional cases

Eigenfunctions in AdS₂

In AdS₂ the scalar curvature term in (3.1) vanishes and the operator (3.1) becomes

$$E^2 = -(1-x^2)\partial_x^2 + x\partial_x + \frac{E_0(E_0-1)}{1-x^2}, \quad (\text{A.1})$$

with $x = x_1 \in (-1, 1)$. The ground state

$$\Psi_{E_0}(x) \sim (1-x^2)^{\frac{E_0}{2}}, \quad (\text{A.2})$$

which is annihilated by the lowering operator $Z = J_{10'} - iJ_{10}$, is an eigenfunction of the operator (A.1) with the eigenvalue E_0^2 . The action of the raising operator $Z^* = J_{10'} + iJ_{10}$ on the ground state creates the first excited state $\Psi_{E_1}(x) \sim x(1-x^2)^{\frac{E_0}{2}}$, which is an eigenfunction of the operator (A.1) with the eigenvalue $(E_0+1)^2$. Continuing this process step by step, with the notation $\psi_k = \Psi_{E_k}$ and $\alpha = E_0$, one finds

$$\psi_{k+1}(x) \sim [(\alpha+k)x - (1-x^2)\partial_x]\psi_k(x), \quad (\text{A.3})$$

and therefore the eigenfunctions can be written in the form

$$\psi_k(x) = c_{k,\alpha} (1-x^2)^{\frac{\alpha}{2}} C_k^\alpha(x), \quad (\text{A.4})$$

where $C_k^\alpha(x)$ is a polynomial of order k and $c_{k,\alpha}$ denotes a normalization constant. According to (A.3), the functions $C_k^\alpha(x)$ satisfy the recursive relations

$$C_{k+1}^\alpha(x) = (2\alpha+k)x C_k^\alpha(x) - (1-x^2)\frac{d}{dx}C_k^\alpha(x), \quad (\text{A.5})$$

of the Gegenbauer Polynomials. These functions are orthogonal to each other under the scalar product with the weight $(1-x^2)^{\alpha-1/2}$ and the normalization coefficients are given by

$$c_{k,\alpha} = 2^\alpha \Gamma(\alpha) \left[\frac{(k+\alpha)k!}{2\pi\Gamma(2\alpha+k)} \right]^{\frac{1}{2}}. \quad (\text{A.6})$$

Using the recursive relation of the Gegenbauer polynomials, one can calculate the matrix elements of the symmetry generators in the basis (A.4) and find

$$\langle \psi_l | Z | \psi_k \rangle = \sqrt{k(k-1+2\alpha)} \delta_{l,k-1}, \quad \langle \psi_l | Z^* | \psi_k \rangle = \sqrt{(k+1)(k+2\alpha)} \delta_{l,k+1}. \quad (\text{A.7})$$

These matrix elements and the spectrum $E_k = \alpha + k$ correspond to the well known unitary irreducible representation of the $\mathfrak{so}(2, 1)$ algebra with the Casimir invariant $C = \alpha(\alpha-1)$.

Note that the parameterization $x = -\cos \sigma$, $\sigma \in (0, \pi)$, gives to (A.1) the form of the Schrödinger operator with the Pöschl-Teller potential [19]

$$E^2 = -\partial_\sigma^2 + \frac{\alpha(\alpha-1)}{\sin^2 \sigma}. \quad (\text{A.8})$$

Eigenfunctions in AdS_3

The symmetry generators in AdS_3 realize the decomposition $\mathfrak{so}(2, 2) = \mathfrak{so}(2, 1) \oplus \mathfrak{so}(2, 1)$ in terms of the left and right operators

$$E_L = E - J_{12} , \quad Z_L = Z_1 - iZ_2 ; \quad E_R = E + J_{12} , \quad Z_R^* = Z_1 + iZ_2 . \quad (\text{A.9})$$

This means that the triplets (E_L, Z_L, Z_L^*) and (E_R, Z_R, Z_R^*) both form a copy of the $\mathfrak{so}(2, 1)$ algebra and the left operators commute with the right ones. Therefore, the states $\Psi_{lr} = (Z_L^*)^l (Z_R^*)^r \Psi_{E_0}$ with different pairs (l, r) are orthogonal to each other.

In the complex coordinates on the disk $\zeta = x_1 + ix_2$, $\bar{\zeta} = x_1 - ix_2$ the operator (3.1) becomes

$$E^2 = \zeta^2 \partial_{\zeta\zeta}^2 + \bar{\zeta}^2 \partial_{\bar{\zeta}\bar{\zeta}}^2 - (4 - 2\zeta\bar{\zeta})\partial_{\zeta\bar{\zeta}}^2 + 2(\zeta\partial_{\zeta} + \bar{\zeta}\partial_{\bar{\zeta}}) + \frac{1}{4} + \frac{(E_0 - 1)^2 - \frac{1}{4}}{1 - \zeta\bar{\zeta}} , \quad (\text{A.10})$$

and the corresponding vacuum wave function (3.11) is

$$\Psi_{E_0}(\zeta, \bar{\zeta}) \sim (1 - \zeta\bar{\zeta})^{\frac{E_0}{2} - \frac{1}{4}} . \quad (\text{A.11})$$

The creation operator Z_L^* can be written as

$$Z_L^* = \sqrt{E} \left(\zeta(E + 1/2 + \zeta\partial_{\zeta} + \bar{\zeta}\partial_{\bar{\zeta}}) - 2\bar{\zeta} \right) \frac{1}{\sqrt{E}} , \quad (\text{A.12})$$

and Z_R^* is obtained by complex conjugation of (A.12). Calculating the action of these raising operators on the eigenfunctions of energy, one can neglect the \sqrt{E} factors in (A.12), since they become numbers. This remark simplifies the recursive relations between the wave functions and one finds

$$\Psi_{lr}(\zeta, \bar{\zeta}) \sim \Psi_{E_0}(\zeta, \bar{\zeta}) \sum_j^{\min(l,r)} (-1)^j \frac{\Gamma(E_0 + l + r - j)}{j! (l-j)! (r-j)!} \zeta^{l-j} \bar{\zeta}^{r-j} , \quad (\text{A.13})$$

which is an eigenfunction of the operator (A.10) with the eigenvalue $(E_0 + l + r)^2$.

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